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PYTHAGOREAN THEOREM IN UNITARY SPACES

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Dedicated to the memory of Professor Dragoslav S. Mitrinović

Many properties of right triangles have been generalized to unitary spaces of dimension n, sometimes in disguise. In this article, a generalization of Pythagoras' Theorem related to areas and volumes is presented.

1. NOTATION

E_n denotes an n-dimensional unitary space and $\mathbf{R}_n$ an n-dimensional Euclidean space. Vectors are denoted by greek letters $\alpha, \beta, \dots$, and scalars by latin letters. The inner product of ξ and η will be (ξ, η), and the norm of ξ is defined by $\|\xi\| = (\xi,\xi)^{1/2}$. $\vec{0}$ indicates the zero vector. Well-known ideas of linear spaces will be assumed. Other definitions will be presented as needed.

2. RIGHT PYRAMIDS IN $\mathbf{R}_3$

A right pyramid in $\mathbf{R}_3$ is a tetrahedron for which all three angles at one vertex are right angles. The face opposite this vertex is called the surface-hypotenuse. In the language of vectors, one can define this as follows.

Let $\{\xi, \eta, \zeta\}$ be a set of non-zero orthogonal vectors in $\mathbf{R}_3$. The convex hull of $\{\vec{0}, \xi, \eta, \zeta\}$ is a right pyramid whose vertex is at $\vec{0}$ (Figure 1). The triangle whose vertices are the endpoints of ξ, η, and ζ is called the *surface-hypotenuse*.

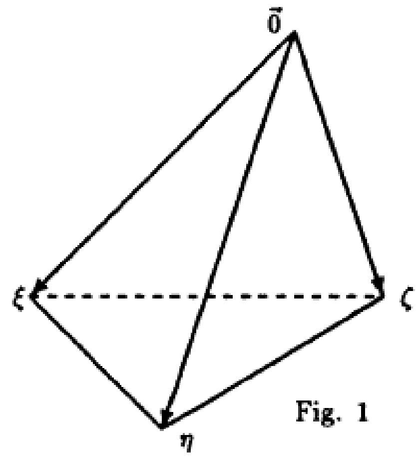

Fig. 1

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Theorem 1. *In a right pyramid the square of the area of the surface-hypotenuse is equal to the sum of the squares of the areas of the other faces.*
Proof. The sum of the squares of the three faces in Figure 1 is

$$\mathcal{A}^2 = \frac{1}{4}\left(\|\xi\|^2\|\eta\|^2 + \|\eta\|^2\|\zeta\|^2 + \|\zeta\|^2\|\xi\|^2\right). \tag{1}$$

The area of the surface-hypotenuse is one-half the area of the parallelogram formed by the vectors $\xi - \zeta$ and $\xi - \eta$. It is well-known that the square of this area is

$$\mathcal{B}^2 = \frac{1}{4}\det\begin{pmatrix} (\xi-\eta,\xi-\eta) & (\xi-\eta,\xi-\zeta) \\ (\xi-\eta,\xi-\zeta) & (\xi-\zeta,\xi-\zeta) \end{pmatrix}. \tag{2}$$

Since $(\xi,\eta) = (\xi,\zeta) = (\eta,\zeta) = 0$, this simplifies to

$$\mathcal{B}^2 = \frac{1}{4}\det\begin{pmatrix} \|\xi\|^2 + \|\eta\|^2 & \|\xi\|^2 \\ \|\xi\|^2 & \|\xi\|^2 + \|\zeta\|^2 \end{pmatrix}. \tag{3}$$

So

$$\mathcal{B}^2 = \frac{1}{4}\left(\|\xi\|^2\|\eta\|^2 + \|\eta\|^2\|\zeta\|^2 + \|\zeta\|^2\|\xi\|^2\right) = \mathcal{A}^2.$$

3. GENERALIZATION TO n DIMENSIONS

Although one can generalize Pythagoras' Theorem in many ways, the approach of Section 2 seems quite natural.

The right pyramid in $\mathbf{R}_n$ is the convex hull of $\{\vec{0},\xi_1,\ldots,\xi_n\}$, where $\{\xi_1,\ldots,\xi_n\}$, is an orthogonal set of non-zero vectors in $\mathbf{R}_n$. The convex hull of the endpoints $\xi_1,\ldots,\xi_n$ is called the *hyperhypotenuse* of the right pyramid. The convex hull of $\{\vec{0},\xi_1,\ldots,\widehat{\xi_i},\ldots,\xi_n\}$, is caled a *hyperface* of the pyramid, where $\widehat{\xi_k}$, $k = 1,\ldots,n$ indicates that element is left out of the sequence.

Theorem 2. *Let $\xi_1,\ldots,\xi_n$ be an orthogonal set of vectore in $\mathbf{R}_n$ forming the edges of a right pyramid. Let $\mathcal{V}$ be the volume of the hyperhypotenuse of the right pyramid, and let $\mathcal{V}_k$ be the volume of the hyperface which is the convex hull of $\{\vec{0},\xi_1,\ldots,\widehat{\xi_k},\ldots,\xi_n\}$. Then*

$$\mathcal{V}^2 = \sum_{k=1}^{n} {\mathcal{V}_k}^2. \tag{4}$$

Proof. As before, we have

$$\sum_{k=1}^{n} {\mathcal{V}_k}^2 = \left(\frac{1}{(n-1)!}\right)^2 \sum_{k=1}^{n}\left(\|\xi_1\|^2\cdots\|\widehat{\xi_k}\|^2\cdots\|\xi_n\|^2\right) \tag{5}$$

and

$$(6)\quad \mathcal{V}^2 = \left(\frac{1}{(n-1)!}\right)^2 \det \begin{pmatrix} (\xi_1-\xi_2,\xi_1-\xi_2) & \dots & (\xi_1-\xi_2,\xi_1-\xi_n) \\ \vdots & & \vdots \\ (\xi_1-\xi_n,\xi_1-\xi_2) & \dots & (\xi_1-\xi_n,\xi_1-\xi_n) \end{pmatrix}$$

It will be important to note later that the determinant in equation (6) is unchanged under permutations of the edges $\xi_1,\dots,\xi_n$. This is clear geometrically because the area of the hyperhypotenuse does not depend on the order in which the edges are given.

By the orthogonality of $\xi_1,\dots,\xi_n$ equation (6) becomes

$$(7)\quad \mathcal{V}^2 = \left(\frac{1}{(n-1)!}\right)^2 \det \begin{pmatrix} \|\xi_1\|^2+\|\xi_2\|^2 & \|\xi_1\|^2 & \dots & \|\xi_1\|^2 \\ \vdots & & & \vdots \\ \|\xi_1\|^2 & & \dots & \|\xi_1\|^2+\|\xi_n\|^2 \end{pmatrix}.$$

Denoting $\|\xi_1\|^2,\dots,\|\xi_n\|^2$ repectively by $a_1,\dots,a_n$, this becomes

$$(8)\quad \mathcal{V}^2 = \left(\frac{1}{(n-1)!}\right)^2 \det \begin{pmatrix} a_1+a_2 & a_1 & \dots & a_1 \\ a_1 & a_1+a_3 & \dots & a_1 \\ \vdots & & & \vdots \\ a_1 & & \dots & a_1+a_n \end{pmatrix}.$$

Let $f(a_1,\dots,a_n)$ denote the determinant in equation (8). It will suffice to prove the identity

$$(9)\quad f(a_1,\dots,a_n) = \sum_{k=1}^{n}(a_1a_2\cdots\widehat{a_k}\cdots a_n).$$

Although this could be proved by a direct but somewhat messy inductive argument, it will turn out to be much simpler to give an argument based on symmetry.

It is clear from its form that $f(a_1,\dots,a_n)$ is a polynomial of degree at most $n-1$ in the variables $a_1,\dots,a_n$. Since the determinant in equation (6) is unchanged under permutation of the edges $\xi_1,\xi_2,\dots,$ f is in fact a *symmetric polynomial*, i.e, $f(a_1,\dots,a_n) = f(a_{\sigma(1)},\dots,a_{\sigma(n)})$ for any permutation σ of $\{1,\dots,n\}$.

Symmetry will be used to calculate the terms of f. by Setting $a_1 = 0$, the terms not involving a_1 will be obtained:

$$(10)\quad f(0,a_2,\dots,a_n) = \det \begin{pmatrix} a_2 & 0 & \dots & 0 \\ 0 & a_3 & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & & a_n \end{pmatrix} = a_2\cdots a_n = \widehat{a_1}a_2\cdots a_n.$$

For any term of a symmetric polynomial, the distinct terms obtained from it by a permutation of the variables are also present. So

$$(11)\quad f(a_1,\dots,a_n) = \sum_{k=1}^{n}(a_1\cdots\widehat{a_k}\cdots a_n) + g(a_1,\dots,a_n),$$

where g, being the difference of two symmetric polynomials, is also a symmetric polynomial. Every non-zero term of g must contain a_1, since we have already found the only term of f that doesn't. By symmetry, every non-zero term of g must contain $a_2, \ldots, a_n$ as well. Thus, every non-zero term of g must have degree at least n, which is impossible. Hence, g is identically 0, and

$$f(a_1, \ldots, a_n) = \sum_{k=1}^{n} (a_1 \cdots \widehat{a_k} \cdots a_n) \tag{12}$$

as required.

An analogous theorem is true for $\xi_1, \ldots, \xi_n$ element of the unitary space E_n. The proof, which is nearly identical, is omitted.

Between the usual formulation of Pythagoras' Theorem

$$\left\| \sum_{k=1}^{n} \xi_k \right\|^2 = \sum_{k=1}^{n} \|\xi\|^2 \tag{13}$$

and the generalization just given in Theorem 2 are a number of other equalities. Their formulations will be left to the reader.

4. ADJOINT PRODUCTS

Let A and B be linear transformations on E_n. Recall that the adjoint A^* is defined by $(A\xi, \eta) = (\xi, A^*\eta)$ for every $\xi, \eta \in E_n$ ([1], [2]).

A^*B is defined to be the *adjoint product* of A and B. If $A^*B = 0$, then A is said to be *adjoint orthogonal* to B. It is clear that $A^*B = 0$ implies $B^*A = 0$.

The theorem of Section 3 can be generalized to the HILBERT norm of a set of adjoint orthogonal linear transformations on E_n. An outline of this will be presented.

Definition 3. *Let $\{A_1, \ldots, A_n\}$ be a set of non-zero orthogonal linear transformations on E_n. The hyperhypotenuse of this set is defined by*

$$\mathcal{S} = \{A_1 - A_2, A_1 - A_3, \ldots, A_1 - A_n\}, \tag{14}$$

and the square of the norm of $\mathcal{S}$ is defined by

(15)
$$\mathcal{N} = \max_{\|\xi\|=1} \det \begin{pmatrix} ((A_1 - A_2)\xi, (A_1 - A_2)\xi) & \ldots & ((A_1 - A_2)\xi, (A_1 - A_n)\xi) \\ \vdots & & \vdots \\ ((A_1 - A_n)\xi, (A_1 - A_2)\xi) & \ldots & ((A_1 - A_n)\xi, (A_1 - A_n)\xi) \end{pmatrix}.$$

The hypersurfaces of the set are $\{A_1, \ldots, \widehat{A_k}, \ldots, A_n\}$. The square of the norm of each hypersurface is defined as

$$\mathcal{N}_k = \max_{\|\xi\|=1} \left\{ \frac{(A_1\xi, A_1\xi) \cdot (A_2\xi, A_2\xi) \cdots (A_n\xi, A_n\xi)}{(A_k\xi, A_k\xi)} \right\}, \quad k = 1, \ldots, n. \tag{16}$$

Lemma 4. *Let λ_k be the square of the Hilbert norm of A_k for $k = 1, \dots, n$. Then*

$$\mathcal{N}_k = \frac{\lambda_1 \cdots \lambda_n}{\lambda_k}. \tag{17}$$

Theorem 5. *Let $\{A_1, \dots, A_n\}$ be an adjoint orthogonal set of linear transformations on E_n. Then*

$$\mathcal{N} = (\lambda_1 \cdots \lambda_n) \sum_{i=1}^{n} \frac{1}{\lambda_i}. \tag{18}$$

Proof. It is observed that

$$\det \begin{pmatrix} ((A_1 - A_2)\xi, (A_1 - A_2)\xi) & \dots & ((A_1 - A_2)\xi, (A_1 - A_n)\xi) \\ \vdots & & \vdots \\ ((A_1 - A_n)\xi, (A_1 - A_2)\xi) & \dots & ((A_1 - A_n)\xi, (A_1 - A_n)\xi) \end{pmatrix} \tag{19}$$

$$= \det \begin{pmatrix} \|A_1\xi\|^2 + \|A_2\xi\|^2 & \dots & \|A_1\xi\|^2 \\ \vdots & & \vdots \\ \|A_1\xi\|^2 & \dots & \|A_1\xi\|^2 + \|A_n\xi\|^2 \end{pmatrix}.$$

Comparing equation (19) with equation (7) and applying the method of Theorem 2, the proof is easily completed.

Again, this result may be generalized in many ways.

REFERENCES

1. A. R. AMIR-MOÉZ, FASS: *Elements of Linear spaces.* Pergamon Press, Oxford, 1962.
2. A. R. AMIR-MOÉZ: *Extreme Properties of Linear Transformations.* Polygonal Publishing House, Washington, N.J. 07882, 1990.

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